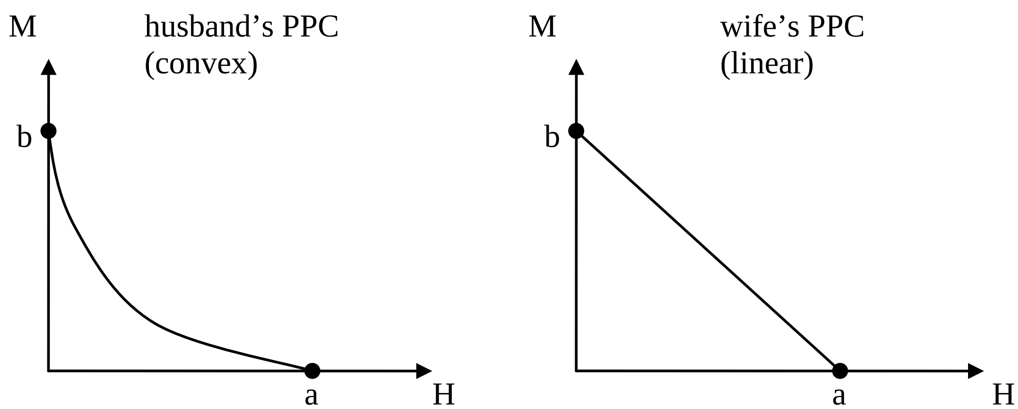
Economics 451 Exam 1 Spring 2016 Prof Montgomery

Answer all questions. 100 points possible.

1) [24 points] Consider a household composed of a husband and a wife. Each member of the household allocates his/her time to produce market goods (M) and household goods (H), and time inputs are chosen to maximize overall household utility (which depends on total M and total H produced by the household members). Assuming that the husband has *increasing* returns to production in each sector, while the wife has *constant* returns to production in each sector, their individual production possibilities curves (PPCs) are given below. Further assume that the horizontal intercepts (labeled a) and the vertical intercepts (labeled b) are the same on both diagrams.



a) In Chapter 2 of Becker's *Treatise*, he considers the case where *both* members of the household have *constant* returns to production in both sectors. Given this assumption, what result does he state regarding the optimal division of labor within households? [NOTE: I'm asking about his initial result, where coefficients of production are taken as given, *not* his subsequent result when individuals choose to invest in human capital.]

b) Given the individual PPCs shown above, draw the household's overall PPC. Then, for each segment of this curve, indicate which members of the household are working in which sector(s). [HINT: Your graph doesn't need to be perfect, but should be properly labeled, and the overall PPC must have the correct shape.] Does Becker's result (discussed in part a) hold in this case?

2) [16 points] Galor argues that rising household income cannot explain the fertility transition. Briefly explain his theoretical claim using equations and/or a graph.

3) [30 points] Consider a marriage market with 4 men and 4 women. The following matrix gives the payoffs received by each man and woman in each potential match. [The matrix specifies the payoffs (m_{ij}, f_{ji}) given a match between man i and woman j. The first column gives payoffs if each man remains single; the first row gives payoffs if each woman remains single.] Assume that utility is *not* transferable.

			women 1	women 2	women 3	women 4
		.	2	3	4	3
men	1	5	(4,5)	(6,9)	(2,7)	(7,4)
men	2	2	(6,4)	(5,1)	(4,5)	(9,1)
men	3	4	(8,3)	(6,2)	(7,3)	(5,2)
men	4	1	(3,6)	(5,5)	(9,2)	(4,6)

a) Is the match structure {M1-F4, M2-F3, M3-F1, M4-F2} stable? If so, state the condition for stability. If not, give one violation of the stability condition.

b) Use the Gale-Shapley algorithm to find two stable match structures. In what sense can these structures be "ranked"? Briefly discuss.

c) Suppose that woman 2 received more utility when single. In particular, suppose now that f_{20} = 7 instead of 3. All other payoffs remain as above. What stable match structure(s) are now generated by the Gale-Shapley algorithm? Is this answer different from your answer to part (b)? Briefly explain why or why not.

4) [30 points] Consider a household with an altruistic wife and selfish husband. The wife's utility is equal to $\ln(Z_W) + (1/2)\ln(Z_H)$ where ln denotes natural logarithm, Z_W is the wife's consumption, and Z_H is the husband's consumption. The wife can transfer some of her income to the husband, but cannot enforce negative transfers.

a) Suppose the household's initial endowment is given by the income levels $I_W = 500$ and $I_H = 100$. How large of a transfer would the wife optimally choose? What are consumption levels for the wife and husband after this transfer is made?

b) The husband is offered a new job in a different city, and can decide unilaterally (without the wife's consent) whether to move the family. If the couple moves, both income levels will be affected. For each case below, I have given the new income levels after the move. For each case, compute the transfer from the wife to the husband and the new consumption levels if this move occurs. Then, for each case, comparing the outcome to the status quo in part (a), indicate whether the husband would accept.

i) $I_W = 600$, $I_H = 60$ ii) $I_W = 300$, $I_H = 120$ iii) $I_W = 180$, $I_H = 240$

c) Explain how your results in part (b) illustrate the Rotten Kid Theorem.

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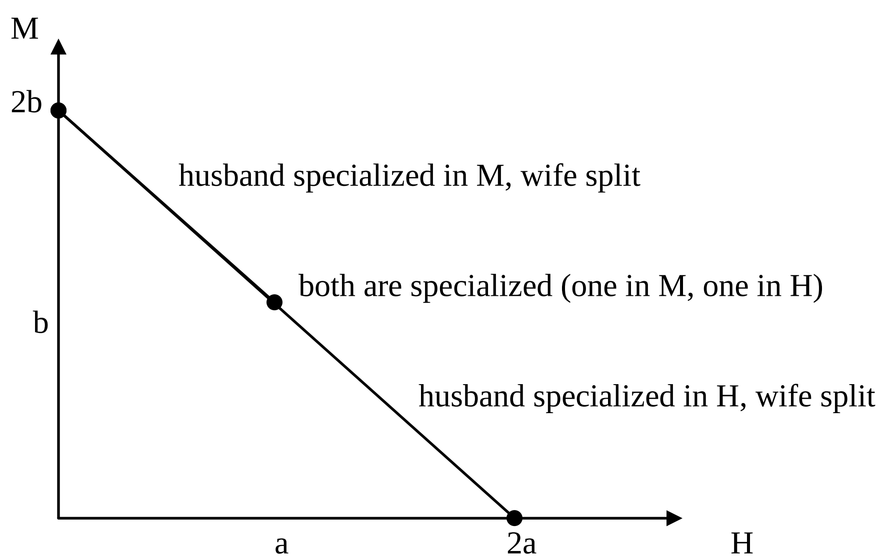
1a) [6 pts] Assuming that household members have different comparative advantages (i.e., their individual PPCs have different slopes), at most one household member should be working in both sectors. Thus, in a two-person household, at least one member of the household should be completely specialized.

1b) [18 pts] Intuitively, it may be more efficient for someone with increasing returns in production to completely specialize (working entirely in the market or entirely in the household) because splitting time between sectors would result in low production in both sectors. In the present example, suppose that both the husband and wife initially allocate all of their time to the market (so that M = 2b and H = 0). If they would like to consume at least some of the household good, it would be more efficient for the wife to start working in the household while the husband remains specialized in the market. (At the vertical intercepts of their PPCs, the wife's PPC is flatter than the husband's PPC, implying that her shift into the household would cause a smaller loss in market production.) Alternatively, suppose that both the husband and wife initially allocate all of their time to the household (so that M = 0 and H = 2a). If they would like to consume at least some of the market good, it would be more efficient for the wife to start working in the market while the husband remains specialized in the household. (At the horizontal intercepts of their PPCs, the wife's PPC is steeper than the husband's PPC, implying that the wife's shift into the market would cause a smaller loss in household production.) Thus, starting from either corner of the HH PPC, the wife would be the member of the household to split her time between sectors. As the wife moves along her individual PPC, reallocating time between sectors, the household PPC must have the same slope as the wife's PPC. Thus, the overall PPC is linear, with the same slope as the wife's individual PPC. The division of labor along each segment is shown below.



Note that Becker's result (from part a) continues to hold: at most one person is split between sectors. (In fact, we obtain the stronger result that the person with increasing returns to production will never split his time between sectors.)

2) [16 pts] Galor assumes that households face the following problem:

maximize $(1-\gamma) \ln c + \gamma \ln n$

subject to $c = y(1-\tau n)$

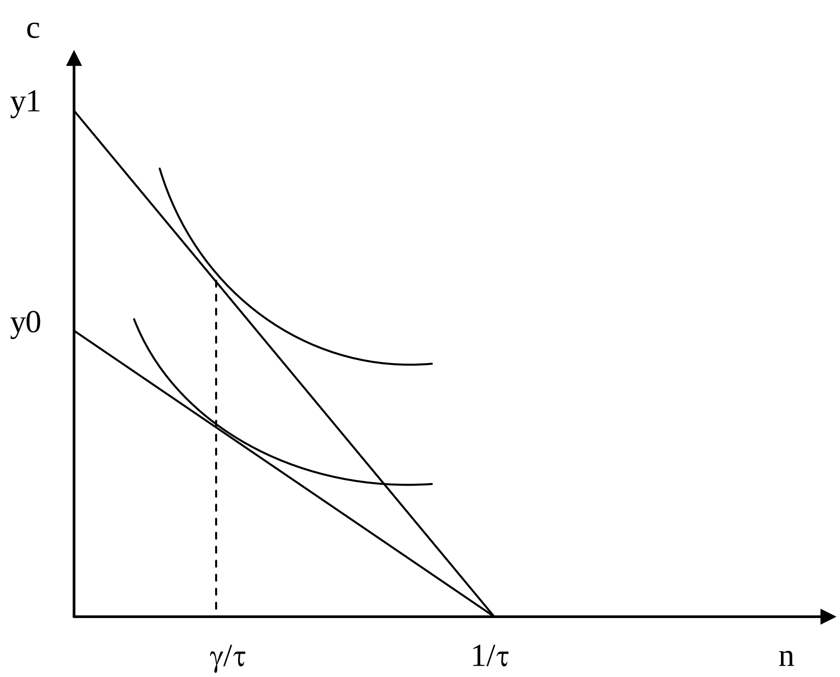
where c = parent's consumption
n = number of children
y = potential HH income (if no children)
τ = cost per child
γ = weight on children in HH utility

Note the particular specification of the utility function (equivalent to Cobb-Douglas) and the particular specification of the constraint (which implies that children impose a "time cost" on parents). Solving this problem, we find that the optimum number of children

$$n^* = \gamma/\tau$$

does not depend on the HH's potential income y. Thus, Galor argues that rising household income (increase in y) cannot explain the fertility transition (decrease in n).

Graphically, an increase in y (from y0 to y1) causes the HH's budget constraint to rotate outwards as shown below. (Essentially, an increase in potential income can be viewed as a reduction in the price of the parent's consumption.) Given the specification of the utility function, the substitution effect (leading to a reduction in the number of children) exactly offsets the income effect (leading to an increase in the number of children).



3a) [5 pts] Not stable. M2 and F1 would prefer each other to their current partners. To see this, note that $m_{21} = 6 > m_{23} = 4$ and $f_{12} = 4 > f_{13} = 3$.

3b) [15 pts] If the men make offers, the Gale-Shapley algorithm generates the stable match structure

{M1-F4, M2-F1, M4-F2, M3 and F3 single}.

If the women make the offers, this algorithm generates the stable match structure

{M1-F2, M2-F1, M4-F4, M3 and F3 single}.

When the men make the offers, the resulting match structure is the stable match structure most preferred by all the men. When the women make the offers, the resulting match structure is the stable match structure most preferred by all the women. To see this, compare the payoffs across the match structures:

	first structure	second structure	
M1	7	6	
M2	6	6	
M3	4	4	all men (weakly) prefer first match structure
M4	5	4	
F1	4	4	
F2	5	9	
F3	4	4	all women (weakly) prefer second structure
F4	4	6	

3c) [10 pts] Given this change in F2's payoffs, the Gale-Shapley algorithm now converges to the same stable match regardless of which side (men or women) makes the offers. The unique match structure is the structure generated in part (b) when the women chose: {M1-F2, M2-F1, M4-F4, M3 and F3 single}. The other structure is no longer stable because F2 would now prefer to remain single than marry M4.

4) In general (without choosing particular numerical values for the income levels), assuming that the wife's altruism is effective, the wife chooses Z_H to maximize

$$\ln(I_W + I_H - Z_H) + (1/2)\ln(Z_H).$$

Differentiating with respect to Z_H and setting this derivative to zero, we obtain

$$[1/(I_W + I_H - Z_H^*)]\,(-1) + (1/2)\,[1/Z_H^*] = 0$$

where Z_H^* is the husband's optimal consumption level (after the transfer from the wife). Solving this equation for Z_H^*, we obtain

$$Z_H^* = (1/3)(I_W + I_H)$$

and $$Z_W^* = (I_W + I_H) - Z_H^* = (2/3)(I_W + I_H)$$

In words, the wife sets the transfer ($y^* = Z_H^* - I_H$) so that the husband's consumption equals 1/3 of the overall household income, and her consumption equals 2/3 of the overall household income. However, it is important to recognize that the preceding solution *assumes* that altruism is effective. If the husband's income I_H exceeds 1/3 of household income, then this solution is not feasible (because negative transfers are not allowed). In this case, the wife would give no transfer, and both members of the household would consume their own initial incomes. Having derived the general solution, it is now straightforward to consider the special cases below.

a) [6 pts] Total household income is 600. Thus, the wife would set the transfer so that the husband's consumption is (1/3)(600) = 200 and her consumption is (2/3)(600) = 400. The transfer is thus 200-100 = 100.

bi) [6 pts] Total household income is 660. After the transfer from the wife, the husband's consumption will be (1/3)(660) = 220 and the wife's consumption will be (2/3)(660) = 440. Thus, the transfer is 220-60 = 160 (which is positive, and thus this solution is feasible). Because the husband's consumption in part a (= 200) is lower than 220, he would accept the new job.

ii) [6 pts] Total household income is 420. After the transfer from the wife, the husband's consumption will be (1/3)(420) = 140 and the wife's consumption will be (2/3)(420) = 280. Thus, the transfer is 140-120 = 20 (which is positive, and thus this solution is feasible). Because the husband's consumption in part a (= 200) is higher than 140, he would NOT accept the new job.

iii) [6 pts] Total household income is again 420. Thus, the solution to the wife's optimization problem again implies that the husband's consumption will be (1/3)(420) = 140 and the wife's consumption will be (2/3)(420) = 280. However, this solution is NOT feasible because it implies a negative transfer (140–240 = –100). Thus, the wife would give no transfer, the wife would consume her own income (= 180) and the husband would consume his own income (= 240). Because the husband's consumption in part a (= 200) is lower than 240, he would accept the new job.

c) [6 pts] The Rotten Kid Theorem states that, as long as the household has an altruistic member whose altruism is effective, selfish household members will attempt to maximize overall household income. In part bi, the husband would increase household income by taking the new job, and thus accepts. In part bii, the husband would decrease household income by taking the new job. Because altruism remains effective after the move, the husband does not accept. In part biii, the husband would again decrease household income by taking the new job. But because altruism is not effective after the move, the Rotten Kid Theorem no longer guarantees that selfish household members will maximize household income, and he accepts.

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Answer all 3 questions. 200 points possible. Explanations can be brief.

1) [40 points] A buyer and seller are negotiating the price of a used car. If they reach an agreement, the buyer's utility will be U_B = 100–p and the seller's utility will be U_S = p–40 where p is the price of the car. If they fail to reach an agreement, the buyer will purchase a different car worth 120 at a price of 105, and thus receive utility T_B = 120–105 = 15 while the seller will sell car to an alternative buyer at a price of 75, and thus receive utility T_S = 75–40 = 35.

a) Explain how the Nash bargaining solution can be used to determine the outcome of this negotiation. What price would be negotiated? What utility would the buyer receive? What utility would the seller receive? Draw an efficient-frontier diagram to illustrate the outcome. [Your diagram doesn't need to be perfect, but should be properly labeled, with relevant numerical values indicated.]

b) Suppose the seller's alternative buyer was willing to pay only 45, so that T_S falls to 5 (while T_B remains at 15). How does this change your answers to part (a)? Again report the price and utility levels. Explain (conceptually) why the Nash bargaining solution has changed.

2) [60 pts] Consider the version of the Hotelling model where candidates (as well as voters) have ideal points. Voters are spread evenly across the (one-dimensional) policy continuum, with their ideal points distributed uniformly between x = 0 and x = 1. Voters have symmetric utility functions, so each voter votes for the candidate closest to the voter's ideal point. There are three candidates who choose locations x_1, x_2, and x_3. If two candidates choose the same location, they split the votes received by that location. The candidate with the highest proportion of votes wins the election (and they flip a coin in case of a tie). Candidates 1 and 2 locate non-strategically. Candidate 3 locates strategically to minimize the distance between his own ideal point (at 0.9) and the winning candidate's location. More precisely, the utility for candidate 3 is $U_3(x) = -|0.9 - x|$ where x is the winning candidate's location.

a) Suppose the candidates locate at x_1 = 0.1, x_2 = 0.4, and x_3 = 0.9. What proportion of the vote is received by each candidate? Who wins the election? What utility is received by candidate 3? Is candidate 3 making his best response (given the locations of the other candidates)? If not, what is the best response for candidate 3? What utility will he receive from this best response?

b) Suppose the candidates locate at x_1 = 0.1, x_2 = 0.5, and x_3 = 0.5. What proportion of the vote is received by each candidate? Who wins the election? What utility is received by candidate 3? Is candidate 3 making his best response (given the locations of the other candidates)? If not, what is the best response for candidate 3? What utility will he receive from this best response?

c) Suppose the candidates locate at x_1 = 0.1, x_2 = 0.7, and x_3 = 0.7. What proportion of the vote is received by each candidate? Who wins the election? What utility is received by candidate 3? Is candidate 3 making his best response (given the locations of the other candidates)? If not, what is the best response for candidate 3? What utility will he receive from this best response?

3) [100 points] Five voters (A, B, C, D, E) rank 4 outcomes (w, x, y, z) as shown below

	A	B	C	D	E
rank 1	x	w	z	z	x
2	z	x	x	y	z
3	y	z	y	w	y
4	w	y	w	x	w

a) Use the Condorcet procedure to determine the group (social) preferences. Are group preferences transitive?

b) Use the Borda count procedure to determine the group preferences.

c) Suppose the group choice is determined through agenda voting using amendment procedure (so that two outcomes are compared; the winner is then compared to a third outcome; the winner is then compared to the fourth outcome). Further assume that voters vote sincerely. Is it possible to structure the agenda in a way that outcome z is the winner? Show the agenda (using a “final four” type diagram) or explain why this is not possible.

d) Suppose that outcome y moves up one notch in each voter's ranking, while the relative positions of the other outcomes are unaffected. Thus, rankings are now given by

	A	B	C	D	E
rank 1	x	w	z	y	x
2	y	x	y	z	y
3	z	y	x	w	z
4	w	z	w	x	w

Use the Borda count procedure to determine the group preferences.

e) Which condition in the Arrow Impossibility Theorem is problematic for the Borda count procedure? Explain, using your answers to parts (b) and (d) to illustrate the problem.

f) After some further changes, voter's preferences are given by

	A	B	C	D	E
rank 1	x	w	z	y	x
2	y	y	x	z	y
3	z	z	y	w	z
4	w	x	w	x	w

Use the Condorcet procedure to determine the group preferences. Are group preferences transitive? Which outcome(s) are in the Condorcet set?

g) Given the answer to part (f), suppose the group choice is determined through agenda voting using amendment procedure with the following order of voting: x vs. y; winner vs. w; winner vs. z. Which outcome wins if voters are sincere? Which outcome wins if voters are sophisticated? Use a game tree to explain the outcome under sophisticated voting.

Econ 451 **Exam 2** **Spring 2016** **Solutions**

1a) [25 pts] The Nash bargaining solution maximizes the product $(U_B - T_B)(U_S - T_S)$ where the Us are utility levels and the Ts are threat points. For this problem,

$$(U_B - T_B)(U_S - T_S) = (100 - p - 15)(p - 40 - 35) = (85 - p)(p - 75)$$

Maximizing with respect to p, we obtain

$$85 - 2p + 75 = 0$$

which implies $p = 80$. Thus, $U_B = 20$ and $U_S = 40$.

[To solve this in a different way, you could first find the efficient frontier:

$$U_B + U_S = 100 - p + p - 40 = 60 \quad \text{and thus} \quad U_S = 60 - U_B$$

You could then substitute the efficient frontier into the Nash bargaining solution:

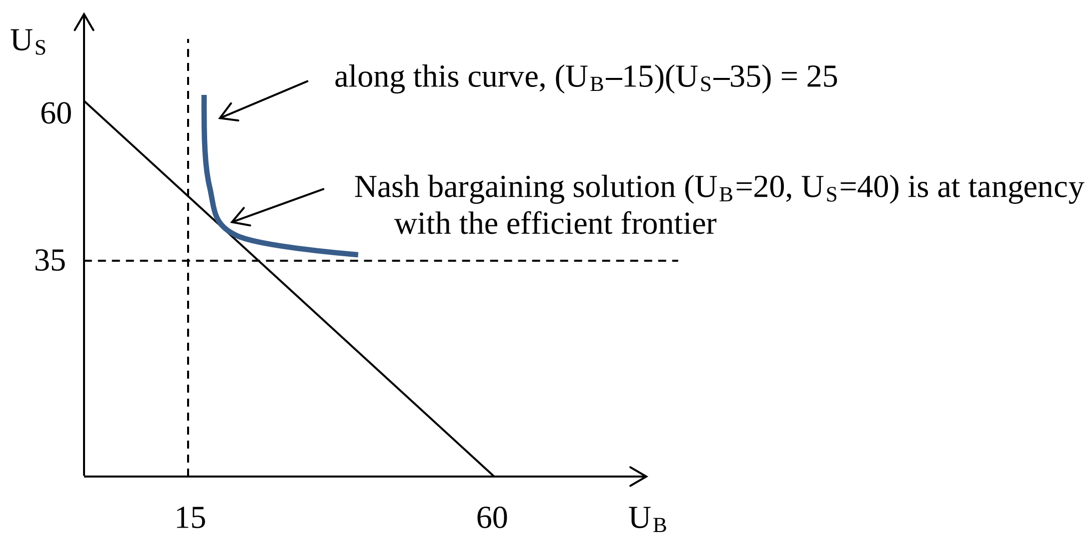
$$(U_B - 15)(60 - U_B - 35) = (U_B - 15)(25 - U_B)$$

Maximizing with respect to U_B, we obtain

$$25 - 2U_B + 15 = 0 \quad \text{which implies } U_B = 20.$$

Thus, $U_S = 40$ and $p = 80$.]

Graphically, using the efficient frontier diagram,



b) [15 pts] Now $(U_B - T_B)(U_S - T_S) = (100 - p - 15)(p - 40 - 5) = (85 - p)(p - 45)$, so the Nash bargaining solution implies $p = 65$, $U_B = 35$, and $U_S = 25$. Conceptually, S's lower threat point weakens her bargaining power, causing the price to fall, the buyer's utility to rise, and the seller's utility to fall.

2a) [20 pts] Candidate 1 receives 25% of the vote (from voters located between x = 0 and x = .25), candidate 2 receives 40% of the vote (from voters located between x = .25 and x = .65), and candidate 3 receives 35% of the vote (from voters located between x = .65 and x = 1). Candidate 2 wins the election. Candidate 3 receives utility $U_3(.4) = -|.9 - .4| = -0.5$. Candidate 3 is not making a best response. He should move leftward just far enough to win the election. More precisely, if candidate 3 moves just to the left of $x_3 = .85$, he will receive just over 37.5% of the vote, while candidate 2 will receive just under 37.5% of the vote. This would allow candidate 3 to win the election, and give him utility just under $U_3(.85) = -0.05$.

b) [20 pts] Candidate 1 receives 30% of the vote (from voters located between x = 0 and x = .3), while the other two candidates receive 35% each (splitting the votes from voters located between x = .3 and x = 1). Either candidate 2 or 3 wins the election (depending on the coin flip); either would enact the same policy once elected (x = .5). Candidate 3 receives utility $U_3(.5) = -0.4$. Candidate 3 is not making a best response. He should move rightward as far as he can while still winning the election. More precisely, if candidate 3 moves just to the left of $x_3 = .8$, he will receive just over 35% over the vote, while candidate 2 will receive just under 35% of the vote. This would allow candidate 3 to win the election, and give him utility just under $U_3(.8) = -0.1$.

c) [20 points] Candidate 1 receives 40% of the vote (from voters located between x = 0 and x = .4), while the other two candidates receive 30% each (splitting the vote from voters located between x = .4 and x = 1). Candidate 1 wins the election. Candidate 3 receives utility $U_3(.1) = -0.8$. Candidate 3 is not making a best response. Given the positions of the other candidates, there is no way for candidate 3 to win the election. However, he could move far enough to the right to allow candidate 2 to win the election. More precisely, if candidate 3 moves to some $x_3 > .9$, he will receive under 10% over the vote, while candidate 2 will receive over 40%. This would make candidate 2 the winner, and candidate 3 would receive utility $U_3(.7) = -0.2$. [Alternatively, candidate 3 could move far enough to the left ($x_3 < .6$) to allow candidate 2 to win the election. While this may seem perverse (given candidate 3's ideal point of 0.9), we've assumed that candidate 3's utility depends only on the winner's position. Thus, candidate 3's position doesn't affect his utility if he doesn't win.]

3a) [15 pts] Using the Condorcet procedure (pairwise comparisons), group preferences are

$x > w$, $y > w$, $z > w$, $x > y$, $x > z$, $z > y$

Group preferences are transitive: $x > z > y > w$.

b) [12 pts] Using the Borda count procedure (assigning points),

w receives 0+3+0+1+0 = 4 points
x receives 3+2+2+0+3 = 10 points
y receives 1+0+1+2+1 = 5 points
z receives 2+1+3+3+2 = 11 points

Group preferences are thus $z > x > y > w$.

c) [9 pts] No, there is no order of voting that would make z the winner. Under amendment procedure, the winner needs to be a member of the Condorcet set. For this example, x is the only outcome in the Condorcet set.

d) [12 pts] The Borda count procedure would now imply

w receives 0+3+0+1+0 = 4 points
x receives 3+2+1+0+3 = 9 points
y receives 2+1+2+3+2 = 10 points
z receives 1+0+3+2+1 = 7 points

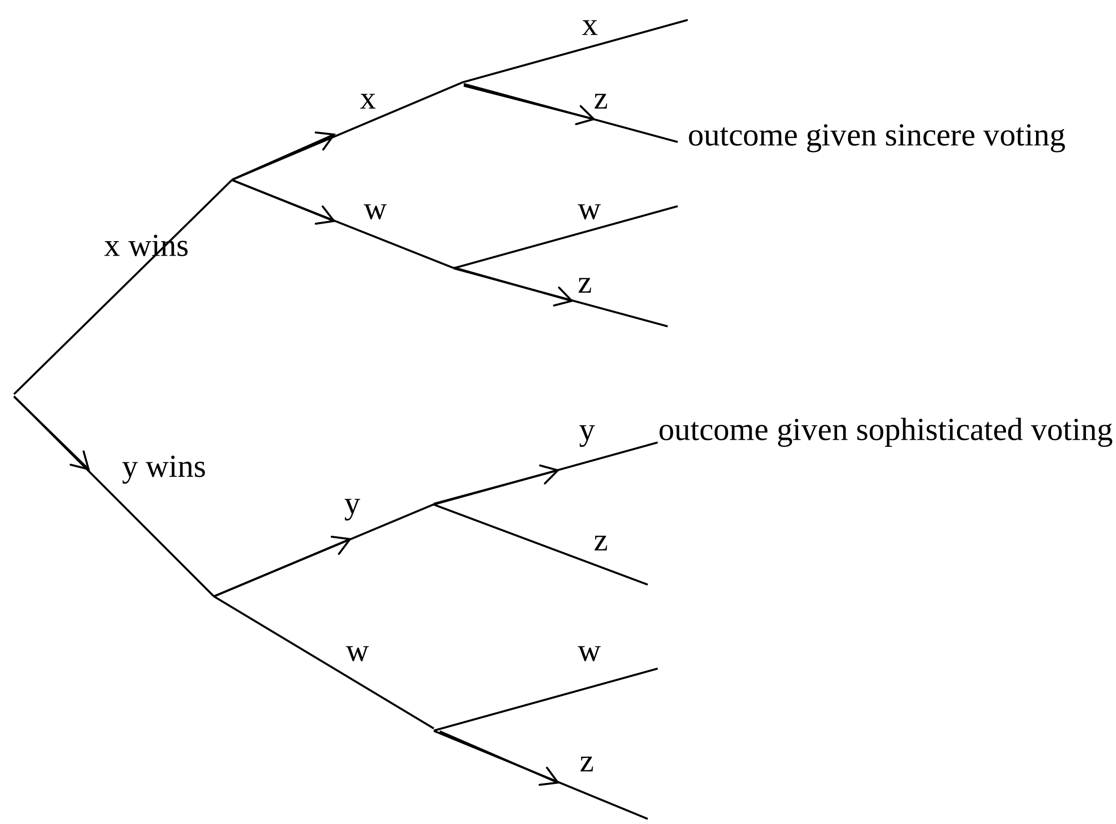
Group preferences are now $y > x > z > w$.

e) [10 pts] The Borda count procedure does not satisfy condition I (Independence of Irrelevant Alternatives). Comparing parts (b) and (d), a change in each voter's ranking of y should have no effect on the relative ranking of x and z. In that sense, y is an "irrelevant alternative." However, as shown, the Borda count procedure implies $z > x$ for part (b), but $x > z$ for part (d).

f) [18 pts] The Condorcet procedure yields $x > w$, $y > w$, $z > w$, $x > y$, $z > x$, $y > z$. Group preferences are not transitive (given the voting cycle from x to y to z to x). The Condorcet set is {x, y, z}.

g) [24 pts] If voting is sincere: x defeats y, then x defeats w, then z defeats x. Thus, z is the winner.

If voters are sophisticated, the following tree shows the possible sequences of voting. The arrows show the outcome at each node (applying backward induction). In the potential contest between x and w, voters would realize that either option would lead to z in the final round, and are thus indifferent between x and w (as indicated by arrows along both branches). In the initial contest between x and y, voters realize that voting for x would ultimately cause z to win, and that voting for y would ultimately cause y to win. Thus, under sophisticated voting, y wins the first round, the second round, and the final round.



Economics 451 Exam 3 Spring 2016 Prof Montgomery

Answer all 4 questions. 130 points possible.

1) [40 points] Bisin and Verdier (*Quarterly Journal of Economics* 2000) develop a model of the socialization process in which individuals acquire either trait A or trait B. Following their paper, let q^i_t denote the proportion of the population with trait i (= A or B) in period t, and note that $q^A_t + q^B_t = 1$ for all t. Traits may be acquired from parents (through vertical socialization) or from other members of the society (through oblique socialization). Each parent has one child, and attempts to instill her own trait in the children. Vertical socialization succeeds with probability τ^A for parents with trait A, and succeeds with probability τ^B for parents with trait B. If vertical socialization fails, then the child adopts the trait of a "cultural parent" selected randomly from the population (i.e., the child adopts trait A with probability q^A_t and adopts trait B with probability q^B_t).

a) Write q^A_{t+1} as a function of q^A_t, τ^A, and τ^B. Assuming that τ^A and τ^B are constants (and that the initial q^A_0 in period 0 is strictly between 0 and 1), what happens in the long run if $\tau^A > \tau^B$? Briefly explain.

b) Now suppose that each parent chooses τ to maximize her expected utility. Each parent with trait A receives utility level V^{AA} if her child acquires trait A, and receives 0 if her child acquires trait B. Similarly, each parent with trait B receives V^{BB} if her child acquires trait B, and receives 0 if her child acquires trait A. Assume that the cost of vertical socialization equals $(c^A)(\tau^A)^2$ for parents with trait A, and equals $(c^B)(\tau^B)^2$ for parents with trait B, where c^A and c^B are constants. Write the expected utility function for a parent with trait A, and then for a parent with trait B. Then solve for the optimal socialization choices (τ^{A*} and τ^{B*}). Briefly discuss how the optimal choice τ^{A*} depends on the constants V^{AA} and c^A.

c) Given the optimal choices τ^{A*} and τ^{B*} from part (b), use the equation from part (a) to solve for the long-run outcome q^A. Assuming that the initial q^A_0 is strictly between 0 and 1, do both traits persist in the long run? Briefly discuss. Then give the numerical solution for the long-run q^A when

i) $V^{AA} = 2$, $V^{BB} = 1$, $c^A = 1$, $c^B = 1$
ii) $V^{AA} = 1$, $V^{BB} = 1$, $c^A = 3$, $c^B = 1$

2) [10 points] Analysis of Pascal's Wager reveals that it is never optimal to participate in a religion with a "forgiving" god. Briefly explain why. To illustrate your answer, construct a version of Pascal's Wager in which an individual chooses between a forgiving god and no religion, and then compare the expected payoffs generated by each choice.

3) [30 points] Consider the version of the Azzi and Ehrenberg (*Journal of Political Economy* 1975) model of church attendance discussed in lecture. In that model, the individual allocates time in order to

maximize $U(z_1) + \beta\, U(z_2) + \beta^2 R(r_1, r_2)$

subject to the budget constraint $z_1 + \beta z_2 = w_1 h_1 + \beta w_2 h_2$

and the time constraints $h_1 + r_1 = T$ and $h_2 + r_2 = T$

where z_t is consumption in period t, r_t is religious participation in period t, w_t is the wage in period t, h_t is hours of work in period t, T is total time per period, β is the discount factor (between 0 and 1), $U(z_t)$ is the utility from consumption in period t, and $R(r_1,r_2)$ is the afterlife reward. To derive their results, Azzi and Ehrenberg further assume that the U and R functions display diminishing returns (i.e., these functions are concave with respect to each input).

3a) What additional assumption did Azzi and Ehrenberg make about the R function? Assuming that wages are constant over time ($w_1 = w_2 = w$), what does the model imply about change in religious participation over the two periods of the individual's life? Briefly discuss this result.

3b) Now suppose that the reward function R may be written as $R(r_1,r_2) = f(r_1) + \beta f(r_2)$ where the f function displays diminishing returns (i.e., f is concave). How does this specification violate the assumption that Azzi and Ehrenberg made about the R function? How would this specification of the R function change the key result (about change in religious participation over time) discussed in part (a)? Verify this formally by solving the individual's optimization problem (deriving the first-order conditions) using the new specification of the reward function.

4) [50 points] Consider a religious group with n members. Suppose that member i's utility is given by

$$(1/n)\ [\ln(r_1) + \ln(r_2) + \ldots + \ln(r_i) + \ldots + \ln(r_n)] + \ln(h_i)$$

where r_j is the religious participation of member j, h_i is the time spent by member i in non-religious activities, and ln( ) is the natural log function. (Note that i's utility thus depends not only on her own participation r_i but also on participation r_j for all $j \neq i$.) Further assume that member i allocates her total time between religious participation and non-religious activities so that $r_i + h_i = T$ where T is total time available.

a) Holding constant the religious participation of the other members, what is member i's optimal solution to this utility maximization problem? Briefly discuss how this solution depends on the number of members (n).

b) Assuming that the other members of the group (j = 1, 2, …, i-1, i+1, …, n) face similar optimization problems and thus make the same optimal choice, what is the utility level received by each member? Is this outcome Pareto optimal? Briefly discuss. What choice would each member make if he/she was attempting to maximize total utility summed over all group members?

c) Suppose that h_i represents time spent at work, and that each hour of work generates one unit of consumption. Further suppose that the religious group is able to destroy some fixed amount of each member's consumption. Formally, suppose that member i's consumption is now $h_i - \gamma$, where γ (set between 0 and T) is the amount of consumption destroyed by the group. Member i's utility thus becomes

$$(1/n)\ [\ln(r_1) + \ln(r_2) + \ldots + \ln(r_i) + \ldots + \ln(r_n)] + \ln(h_i - \gamma).$$

Derive the new optimal solution for member i. Suppose the group increases γ. Will member i increase or decrease religious participation? Will member i's consumption rise or fall? Given that other members face the same optimization problem (and thus derive the same optimal solution), does each member's utility rise or fall when the group increases γ?

d) Briefly discuss Iannaccone's (*American Journal of Sociology* 1994) explanation for why strict churches are "strong." Is the religious group in part (c) correctly implementing Iannaccone's insight? Briefly explain.

Econ 451 Exam 3 Spring 2016 Solutions

1a) [10 pts] In words, the proportion of A's in period t+1

= (proportion of A's in period t) × (probability an A parent has an A child)
+ (proportion of B's in period t) × (probability a B parent has an A child)

Using mathematical notation,

$$q^A_{t+1} = q^A_t\,[\tau^A + (1-\tau^A)q^A_t] + (1-q^A_t)\,[(1-\tau^B)q^A_t]$$

which simplifies (somewhat) to

$$q^A_{t+1} = q^A_t + (\tau^A - \tau^B)\, q^A_t\,(1-q^A_t)\ .$$

Thus, assuming that $\tau^A-\tau^B$ is positive and that q^A_t is strictly between 0 and 1, q^A_{t+1} is always larger than q^A_t, and q^A converges to 1. In the long run, everyone will possess the trait associated with more successful vertical (parent-to-child) socialization.

b) [15 pts] In words, the expected utility of a parent with trait A

= (probability an A parent has an A child) × (A parent's value of an A child)
+ (probability an A parent has a B child) × (A parent's value of a B child)
– cost of socializing child

Thus, $EU_A = [\tau^A + (1-\tau^A)q^A_t]\, V^{AA} + (1-\tau^A)\,(1-q^A_t)\, 0 - (c^A)(\tau^A)^2$.

Similarly, $EU_B = [\tau^B + (1-\tau^B)(1-q^A_t)]\, V^{BB} + (1-\tau^B)\, q^A_t\, 0 - (c^A)(\tau^A)^2$.

To find the optimal socialization choices, we differentiate each expression with respect to the relevant τ, set the derivative to 0, and then solve for τ. For parents with trait A,

$$dEU_A/d\tau^A = (1-q^A_t)V^{AA} - 2\, c^A\, \tau^A = 0$$

and thus $\tau^{A*} = (1-q^A_t)V^{AA} / (2c^A)$.

Similarly, for parents with trait B, we obtain $\tau^{B*} = q^A_t V^{BB} / (2c^B)$.

Intuitively, parents will spend more effort socializing children (set τ^A or τ^B higher) when their value of an own-trait child is higher (V^{AA} or V^{BB}) is higher or when the cost of socializing children (c^A or c^B) is lower.

1c) [15 pts] Substituting τ^{A*} and τ^{B*} into the equation from part (a), and setting $q^A_{t+1} = q^A_t = q^A$ to obtain the long-run outcomes, we obtain

$$[((1-q^A)V^{AA}/(2c^A)) - (q^A V^{BB}/(2c^B))]\, q^A\, (1-q^A) = 0 \ .$$

As discussed in lecture, the extreme outcomes ($q^A = 0$ or 1) are unstable. The unique stable equilibrium is thus given by

$$q^A = c^B V^{AA}/(c^A V^{BB} + c^B V^{AA}).$$

Thus, $q^A = 2/(1+2) = 2/3$ in case (i), and $q^A = 1/(3+1) = 1/4$ in case (ii).

2) [10 pts] A "forgiving" god rewards both followers and non-followers. A simple version of Pascal's Wager with a forgiving god would be

	god exists (probability q)	god does not exist (probability 1-q)
believe	R-C	-C
don't believe	R	0

where there are two possible outcomes (god exists [probability q]; god doesn't exist [probability 1-q]) and two possible choices (believe; don't believe). [As we discussed in lecture, "believe" or "don't believe" should be interpreted as actions, since the individual's subjective beliefs are already reflected by the probability q. Thus, it might be better to label these choices as "religious participation" and "non-participation," or maybe "follow" and "not follow."] Given the payoff structure assumed above, R is an afterlife reward, C is the cost of religious participation, and god is "forgiving" in the sense that both believers and non-believers would receive the reward (conditional on god's existence). Computing the expected utility of each choice, the expected utility of belief is $qR - C$, while the expected utility on non-belief is qR. Assuming that C is positive (i.e., religious participation imposes costs rather than benefits), non-belief is always the better choice (regardless of the probability q).

3a) [10 pts] Azzi and Ehrenberg assumed that religious participation is "equally productive" across time. Formally, if $r_1 = r_2$, then the marginal return to religious participation is assumed equal across periods: $\partial R/\partial r_1 = \partial R/\partial r_2$. Solving the individual's utility maximization problem, we equate marginal gains (from additional religious participation) with marginal losses (from decreased market work) in each period. The optimum levels of religious participation are thus determined by the equations

$$(dU/dz)\, w_1 = \beta^2\, (\partial R/\partial r_1) \qquad \text{in period 1}$$

and $$(dU/dz)\, w_2 = \beta\, (\partial R/\partial r_2) \qquad \text{in period 2.}$$

Assuming that religious participation is "equally productive" across time and that wages are constant across time ($w_1 = w_2 = w$), the crucial difference in these equations is the additional β in the first equation, which implies that optimal r_1 will be lower than optimal r_2. Intuitively, because the "afterlife reward" is two periods away in period 1 and only one period away in period 2, the individual would choose higher religious participation in period 2.

b) [20 pts] Given the reward function $R(r_1, r_2) = f(r_1) + \beta\, f(r_2)$, religious participation is assumed to be *less productive* in period 2. Formally, the marginal gains from an additional hour of religious participation will be

$$\partial R/\partial r_1 = df/dr_1 \qquad \text{in period 1}$$

and $$\partial R/\partial r_2 = \beta\,(df/dr_2) \qquad \text{in period 2.}$$

Substituting into the (marginal gains = marginal losses) equations from part a, we obtain

$$(dU/dz)\, w_1 = \beta^2\,(df/dr_1) \qquad \text{in period 1}$$

and $$(dU/dz)\, w_2 = \beta^2\,(df/dr_2) \qquad \text{in period 2.}$$

Now, if wages are constant across time, optimal religious participation will also be constant across time. Intuitively, the decreased productivity of religious participation in period 2 exactly offsets the time-discounting effect emphasized by Azzi and Ehrenberg.

4a) [10 pts] Substituting the time constraint into the utility function ($h_i = T - r_i$), and then differentiating with respect to r_i, we obtain

$$(1/n)(1/r_i) - 1/(T-r_i) = 0 \qquad \text{which implies} \quad r_i = T/(1+n).$$

Thus, members participate less in larger groups (i.e., r_i is lower when n is larger).

b) [15 pts] If all members face the same optimization problem, then $r_j = T/(1+n)$ for all j (= 1, …, n). Substituting into the utility function, we obtain each member's utility level

$$(1/n)\,(n)\,\ln(T/(1+n)) + \ln(T - (T/(1+n))$$

which can be rewritten as $\ln(T/(1+n)) + \ln(Tn/(1+n))$.

Note that each member allocates a very small proportion of his/her time [1/(1+n)] to religious participation, and a very large proportion of his/her time [n/(1+n)] to non-religious activities. This outcome is not Pareto optimal. Given the positive externalities generated by religious participation, each member spends too little time in religious participation (and too much time in non-religious activities). To derive this result formally, we may choose participation levels to maximize overall (and hence average) utility. Summing utilities over all members, the total utility level for the group is

$$\ln(r_1) + \ln(r_2) + \ldots + \ln(r_n) + \ln(T\text{-}r_1) + \ln(T\text{-}r_2) + \ldots + \ln(T\text{-}r_n).$$

Differentiating with respect to each r_j, we obtain $1/r_j - 1/(T\text{-}r_j) = 0$ for each member j, and thus $r_j = T/2$ for each j. Thus, all members would be better off if each split his/her time equally between religious participation and non-religious activities.

c) [15 pts] Substituting the time constraint into the utility function ($h_i = T - r_i$), and then differentiating with respect to r_i, we obtain

$$(1/n)(1/r_i) - 1/(T - r_i - \gamma) = 0 \quad \text{which implies} \quad r_i = (T\text{-}\gamma)[1/(1+n)]$$

and thus consumption = $T - r_i - \gamma = (T\text{-}\gamma)[n/(1+n)]$

From these equations, we see that the religious group is essentially destroying γ hours of each member's time, while each member continues to devote the same proportion of his/her time to each activity. Thus, as γ rises, each member's religious participation and consumption and utility will fall.

d) [10 pts] Iannaccone argued that, given the positive externalities generated by religious participation, members would choose suboptimal participation levels. To solve this free-rider problem, Iannaccone argued that religious groups will raise the costs (or, equivalently, lower the benefits) of interacting with outsiders. This increases participation within the group and (potentially) makes all members better off.

Given the type of cost imposed by the religious group in part c, the example is not consistent with Iannaccone's argument. (As we saw, increasing γ makes members worse off, not better off.) The moral is that religious groups cannot impose costs on members in arbitrary ways. Rather, costs need to be imposed in such a way that they will increase the individual's optimal level of religious participation. The sort of "lump sum" costs imposed on members in part c do not reduce the marginal benefit of non-religious activities. In fact, these costs actually increase the marginal benefit of non-religious activities by reducing consumption and thus increasing the marginal utility of consumption.